# A Separable Normed Space that is Isometric to a Hamel Base of Itself

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## ABSTRACT

***It is shown that there is a linear subspace of the Banach space of all null real sequences that admits a nonlinear isometry onto some algebraic base of itself.***

**Key words:** Banach space, Hamel base, Nonlinear isometry

## INTRODUCTION

The nonlinear Banach space theory represents today a flourishing field of research and, particularly, questions related to the existence of uniform or Lipschitz homeomorphisms between a Banach space and some subset of another. See for exemple, the beautiful results (Aharoni et al., 1985), concerning the class of Banach spaces that may be embedded into a Hilbert sphere.

The question whether a Banach space can be homeomorphic to a Hamel base of itself was raised by the second author (Duma, 2001). A partial, negative answer was settled (Bartoszynski et al., in press), using the fact that no separable Banach space can have an analytic Hamel basis. Nevertheless, this problem seems to remain unanswered in the non-separable framework.

However, if we restrict ourselves to the normed (not necessarily complete) spaces context, then one can prove a stronger, affirmative result and this will be the main purpose of the present paper. In the sequel, we shall denote by $c_0$ the usual Banach space of all real sequences converging to zero, endowed with its natural sup-norm. For further references and information concerning classical Banach spaces, one may consult Day (1973), Lacey (1974), Lindenstrauss and Tzafriri (1977) and Beauzamy (1985).

## THE MAIN RESULT

***Theorem 1.*** *There exists a linear, infinite-dimensional subspace of $c_0$ that is isometric to a Hamel base of itself.*

The proof of this result, which will be given in the third section, heavily relies on the followings:

***Lemma 2.*** *There is a nonlinear isometry* $T : c_0 \to c_0$ *having linearly independent range.*

**Proof.** Let us define first a one-to-one, nonexpansive mapping $C : c_0 \to c_0$ having linearly independent range.

With this goal in mind, let us consider a bijection

$$k \rightarrow (i_k , j_k)$$

from N* to N* x N* and let us pick a sequence $(W_m)_{m \quad 1} \subset c_0$ that is dense in $c_0$. Next, let us introduce the mapping $C : c_0 \rightarrow c_0$ defined by the formula

$$Cx = \left\{ \frac{1}{k} \cdot \max\left(0 \ \frac{1}{i_k} \ - \ \| x - w_{jk} \|\right)\right\}_{k1} \qquad (x \in c_0) .$$

Clearly, $C$ satesfies $\|Cx - Cy\| \quad \|x - y\|$ for all $x$,$y$ in $c_0$. In order to prove that $C$ is injective and its range is linearly independent, let us take a finite family of distinct vectors $(x_p)_{1 \ p \ n} \subset c_0$ and a corresponding set of scalars $(\lambda_p)_{1 \ p \ n}$ such that

$${}_{p=1}^{n} \ \lambda p C x_p = 0 \ .$$

It follows that ${}_{p=1}^{n} \ \lambda p(Cx_p)_k = 0$ for each $k$ in N* . Let $\delta = \min_{q \neq r} \|x_q - x_r\| > 0$ and let us choose an $i$ in N* so that

$$\frac{1}{i} < \frac{\delta}{3} \ .$$

Since $(W_m)_{m \quad 1}$ is dense, it is possible to find, for any fixed $p^*$, some $j$ in N* with $\|x_{p^*} - w_j\| < \frac{1}{i}$ . Now, let $k$ in N* satisfying $i_k = i$ and $j_k = j$ . Then one get

$$\|x_q - w_j\| \qquad \delta - \frac{1}{i} > \frac{1}{i}$$

for every $q \neq p^*$ and, consequently, $\lambda_{p^*} \cdot \max\ (0, \ \frac{1}{i} \ - \|x_{p^*} - w_j\|) = 0$, from which one obtain $\lambda_{p^*} = 0$.

To end the proof of Lemma, let us define the linear operators $A$,$B$: $c_0 \rightarrow c_0$ by

$$Ax = (x_1,0, x_2,0,x_3,0,...). \text{ and}$$
$$Bx = (0,x_1,0,x_2,0,x_3,...),$$

where $x = (x_1,x_2,x_3,...) \in c_0$.

Finally, we introduce the operator $T : c_0 \rightarrow c_0$ , expressed by $T = A + BC$ , i.e.,

$$Tx = (x_1,(Cx)_1,x_2,(Cx)_2,x_3,(Cx)_3,...).$$

Obviously, we have $\|Tx - Ty\| = \max\ \{\|x - y\| \ , \|Cx - Cy\|\} = \|x - y\|$, hence $T$ is an isometry. Since $C$ is a bijection between $c_0$ and a suitable linearly independent subset, it follows that $T$ has the same property. The Lemma is proved. ■

## PROOF OF THE THEOREM

Because the above built operator $T$ is an isometry, we obtain, as a consequence of the Banach Contraction principle, the existence of some b* $\in c_0$, such that $Tb^* = 2b^*$.Clearly, $b^* \neq 0$.

Let now define the following family of subsets of $c_0$ :

$$F = \{S \subset c_0 \mid S \text{ is a linear subspace; } b^* \in S \text{ and } T(S) \subset S\}.$$

Since $c_0$ belongs to $F$, one may conclude that $F$ is non-empty. Then, it makes sence to define

$$X = \bigcap_{S \in F} S,$$

which evidently is a linear subspace of $c_0$ , containing $b^*$ . Hence, dim $X$  1.   We shall prove that *span* $T(X) = X$ .

STEP 1 : Since X $\subset S$ for each $S$ in $F$, it follows that $T(X) \subset T(S) \subset S$ for each $S$ in $F$, whence *span* $T(X) \subset S$ $(S \in F)$ . Consequently, *span* $T(X) \subset X$.

STEP 2 : Let us denote *span* $T(X)$ by $M$ . Now, from $M \subset X$ one get $T(M) \subset T(X) \subset$ M. It is easy to remark that $b^*$ lies in $M$ , thus $M \in F$ . Then, $X \subset M$ , which, in turn, gives us $X = M$ .

Finally, observe that dim $X = |T(X)|$ must be infinite, because finite-dimensional subspaces have finite bases. This proves the Theorem.

## REFERENCES


Aharoni, I., B. Maurey, and B.S. Mityagin. 1985. Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces. Israel J. Math. 52:251–265.

Bartoszyski, T., M. Dzamonja, L. halbeisen, E. Murtinova, and A. Plichko. On Bases in Banach spaces. Studia Mathematica (in press).

Beauzamy, B. 1985. Introduction to Banach spaces and their geometry. 2nd rev. ed. North-Holland Mathematics Studies, 68. Notas de Matemática (86). Amsterdam, New York, Oxford: North-Holland.

Day, M.M. 1973. Normed linear spaces. 3rd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 21. Berlin-Heidelberg-New York: Springer-Verlag.

Duma, A. 2001. Pathwise connected Hamel bases. Topology Atlas, Open Questions, http://at.yorku.ca.

Lacey, H.E. 1974. The isometric theory of classical Banach spaces. Die Grundlehren der mathematischen Wissenschaften. Band 208. Berlin-Heidelberg-New York: Springer-Verlag.

Lindenstrauss, J., and L. Tzafriri. 1977. Classical Banach spaces I. Sequence spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete. 92. Berlin-Heidelberg-New York: Springer-Verlag.