A Separable Normed Space that is Isometric to a Hamel Base of Itself

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ABSTRACT

It is shown that there is a linear subspace of the Banach space of all null real sequences that admits a nonlinear isometry onto some algebraic base of itself.

Key words: Banach space, Hamel base, Nonlinear isometry

INTRODUCTION

The nonlinear Banach space theory represents today a flourishing field of research and, particularly, questions related to the existence of uniform or Lipschitz homeomorphisms between a Banach space and some subset of another. See for exemple, the beautiful results (Aharoni et al., 1985), concerning the class of Banach spaces that may be embedded into a Hilbert sphere.

The question whether a Banach space can be homeomorphic to a Hamel base of itself was raised by the second author (Duma, 2001). A partial, negative answer was settled (Bartoszynski et al., in press), using the fact that no separable Banach space can have an analytic Hamel basis. Nevertheless, this problem seems to remain unanswered in the non-separable framework.

However, if we restrict ourselves to the normed (not necessarily complete) spaces context, then one can prove a stronger, affirmative result and this will be the main purpose of the present paper. In the sequel, we shall denote by c_0 the usual Banach space of all real sequences converging to zero, endowed with its natural sup-norm. For further references and information concerning classical Banach spaces, one may consult Day (1973), Lacey (1974), Lindenstrauss and Tzafriri (1977) and Beauzamy (1985).

THE MAIN RESULT

Theorem 1. *There exists a linear, infinite-dimensional subspace of c_0 that is isometric to a Hamel base of itself.*

The proof of this result, which will be given in the third section, heavily relies on the followings:

Lemma 2. *There is a nonlinear isometry* $T : c_0 \rightarrow c_0$ *having linearly independent range.*

Proof. Let us define first a one-to-one, nonexpansive mapping $C : c_0 \rightarrow c_0$ having linearly independent range.