On the Weak Solution of the Compound Ultra-hyperbolic Equation

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ABSTRACT

In this paper we have studied the compound ultra-hyperbolic equation of the form

$$\sum_{r=0}^{m} c_r \square^r u(x) = f(x),$$

where $\square^r$ ***is the ultra-hyperbolic operator iterated r-times (r = 0, 1, 2, ..., m), f is a given generalized function, u is an unknown function,*** $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ ***the Euclidean n-dimensional spaces and*** c_r ***is a constant.***

It is found that the equation above has a weak solution u(x) which is of the form Marcel Riesz's kernel and moreover, such a solution is unique.

1. INTRODUCTION

Consider the equation

$$\square^k u(x) = f(x), \tag{1.1}$$

where u and f are some generalized functions, and $\square^k$ is the ultra-hyperbolic operator iterated k -times and is defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.2}$$

$p + q = n$ is the dimension of the space $\mathbb{R}^n$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, and k is a nonnegative integer.

Trione (1987) has shown that (1.1) has $u(x) = R_{2k}(x)$ as a unique elementary solution where $R_{2k}(x)$ is defined by (2.1) with $\alpha = 2k$. Moreover, Tellez (1994) has proved that $R_{2k}(x)$ exists only for case p is odd with $p + q = n$.

In this paper we develop the equation (1.1) to the form

$$\sum_{r=0}^{m} c_r \square^r u(x) = f(x), \tag{1.3}$$

which is called *the compound ultra-hyperbolic equation* and by convention $\square^0 u(x) = u(x)$. We use the method of convolution of tempered distribution to find the solution of equation (1.3).

2. PRELIMINARIES

Definition 2.1 Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$ and write

$$V = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2,\ p + q = n,$$

Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, V > 0\}$, which designates the interior of the forward cone, and $\overline{\Gamma}_+$ designates of its closure, and the following functions introduced by Nozaki (1964) that

$$R_\alpha(x) = \begin{cases} \dfrac{V^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+. \end{cases} \tag{2.1}$$

$R_\alpha(x)$ is called *the ultra-hyperbolic kernel of Marcel Riesz.*

Here α is a complex parameter and n is the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\alpha+2-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)} \tag{2.2}$$

and p is the number of positive terms of

$$V = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2,\ p + q = n, \tag{2.3}$$

and let supp $R_\alpha(x) \subset \overline{\Gamma}_+$.

Now $R_\alpha(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$, and is a distribution of α if $\mathrm{Re}(\alpha) < n$.

Definition 2.2 A generalized function $u(x)$ is called *an elementary solution* of n-dimensional ultra-hyperbolic operator iterated k-times if $u(x)$ satisfies the equation $\Box^k u(x) = \delta$, where $\Box^k$ defined by (1.2) and δ is the Dirac-delta distribution.

Lemma 2.1 $R_\alpha(x)$ is a homogeneous distribution of order $\alpha - n$. In particular, it is a tempered distribution.

Proof. We need to show that $R_\alpha(x)$ satisfies the Euler equation

$$\sum_{i=1}^{n} x_i \frac{\partial^2}{\partial x_i} R_\alpha(x) = (\alpha - n)R_\alpha(x).$$

Now

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_\alpha(x) = \frac{1}{K_n(\alpha)} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2)^{\frac{\alpha-n}{2}}$$

$$= \frac{1}{K_n(\alpha)} (\alpha - n) (x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2)^{\frac{\alpha-n-2}{2}}$$

$$\text{x } (x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2)$$

$$= \frac{1}{K_n(\alpha)} (\alpha - n) (x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2)^{\frac{\alpha-n}{2}}$$

$$= (\alpha - n)R_\alpha(x).$$

Hence $R_\alpha(x)$ is a homogeneous distribution of order $\alpha - n$. Donoghue (1969) proved that every homogeneous distribution is a tempered distribution. So $R_\alpha(x)$ is a tempered distribution. This is complete of proof.

Lemma 2.2 The function $u(x) = R_{2k}(x)$ with $\alpha = 2k$ of (2.1) is the unique elementary solution of the equation $\square^k u(x) = \delta$

Proof. See Trione (1987) and Tellez (1994).

Lemma 2.3 Let $R_\alpha(x)$ and $K_n(\alpha)$ be defined by (2.1) and (2.2). Then
(a) $K_n(\alpha + 2) = \alpha(\alpha + 2 - n) K_n(\alpha)$,
(b) $R_{-2k}(x) = \square^k \delta$, where k is a nonnegative integer,
(c) $\square^k R_\alpha(x) = R_{\alpha-2k}(x)$, where k is a nonnegative integer.

Proof. See Trione (1987).
Moreover, from (b) we obtain $R_0(x) = \delta$ and also from (c)

$$\square^k R_{2k}(x) = R_0(x) = \delta$$

Lemma 2.4 (The convolution of tempered distributions)
(a) $(\square^k \delta) * u(x) = \square^k u(x)$ where u is any tempered distribution.
(b) Let $R_\alpha(x)$ and $R_\beta(x)$ be defined by (2.1) then $R_\alpha(x) * R_\beta(x)$ exists and is a tempered distribution.
(c) Let $R_\alpha(x)$ and $R_\beta(x)$ be defined by (2.1) and if $R_\alpha(x) * R_\beta(x) = \delta$ then $R_\alpha(x)$ is an inverse of $R_\beta(x)$ in the convolution algebra, denoted by $R_\alpha(x) = R_\beta^{*-1}(x)$, moreover $R_\beta^{*-1}(x)$ is unique.

Proof.

(a) First, we consider the case $k = 1$, now

$$\Box\delta = \sum_{i=1}^{p}\frac{\partial^2\delta}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2\delta}{\partial x_j^2},\ p + q = n$$

and let $\varphi(x)$ be a testing function in the Schwartz space s. By the definition of convolution, we have

$$
\begin{aligned}
<(\Box\delta) * u(x), \varphi(x)> &= <u(x), <\Box\delta(y), \varphi(x+y)>> \\
&= <u(x), <\sum_{i=1}^{p}\frac{\partial^2\delta(y)}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2\delta(y)}{\partial x_j^2}, \varphi(x+y)>> \\
&= <u(x), <\delta(y), \sum_{i=1}^{p}\frac{\partial^2\varphi(x+y)}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2\varphi(x+y)}{\partial x_j^2}>> \\
&= <u(x), <\sum_{i=1}^{p}\frac{\partial^2\varphi(x)}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2\varphi(x)}{\partial x_j^2}> \\
&= <\sum_{i=1}^{p}\frac{\partial^2 u(x)}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2 u(x)}{\partial x_j^2}, \varphi(x)> \\
&= <\Box u(x), \varphi(x)>.
\end{aligned}
$$

It follows that $(\Box\delta)*u(x) = \Box u(x)$. Similarly for any k, we can show that $(\Box^k\delta)*u(x) = \Box^k u(x)$.

(b) Since $R_\alpha(x)$ and $R_\beta(x)$ are tempered distributions by Lemma 2.1. Now choose supp $R_\alpha(x) = K \subset \overline{\Gamma}_+$ where K is a compact set and $\overline{\Gamma}_+$ appear in Definition 2.1. Hence, by Donoghue (1969), $R_\alpha(x)*R_\beta(x)$ exists and is tempered distribution.

(c) Since $R_\alpha(x)$ and $R_\beta(x)$ are tempered distributions with compact supports, thus $R_\alpha(x)$ and $R_\beta(x)$ are the elements of space of convolution algebra U' of distribution. Now $R_\alpha(x)* R_\beta(x) = \delta$ then by Zemanain (1965) show that $R_\alpha(x) = R_\beta^{*-1}(x)$ is a unique inverse.

For example, if $\alpha = 2k$ where k is nonnegative integer and by Kananthai (1997), we have $R_{-2k}(x)$ is an inverse of $R_{2k}(x)$, that is

$$R_{2k}(x)*R_{-2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta$$

3. RESULTS

Theorem 3.1 Given the compound ultra-hyperbolic equation

$$\sum_{r=0}^{m} c_r \Box^r u(x) = f(x), \tag{3.1}$$

where $\Box^r$ is the ultra-hyperbolic operator iterated r -times ($r = 0,1,2,\ldots,m$) defined by (1.2), f is a tempered distribution, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ the Euclidean n-dimensional spaces and n is odd and c_r is a constant. Then (3.1) has a unique weak solution

$$u(x) = f(x) * R_{2m}(x) * (c_m R_0(x) + W(x) R_2(x))^{*-1} \tag{3.2}$$

where

$$W(x) = c_{m-1} + c_{m-2} \cdot \frac{V}{2(4-n)} + c_{m-3} \cdot \frac{V^2}{2.4(4-n)(6-n)} + \ldots$$

$$+ c_0 \cdot \frac{V^{m-1}}{2.4.6 \ldots 2(m-1)(4-n)(6-n)\ldots(2m-n)} \tag{3.3}$$

and V defined by (2.3) and $(c_m R_0(x) + W(x) R_2(x))^{*-1}$ is an inverse of $c_m R_0(x) + W(x) R_2(x)$.

Proof. By Lemma 2.4(a), equation (3.1) can be written as

$$(c_m \Box^m \delta + c_{m-1} \Box^{m-1} \delta + \ldots + c_1 \Box \delta + c_0 \delta) * u(x) = f(x)$$

Convolving both sides by $R_{2m}(x)$ defined by (2.1), we obtain

$$(c_m \Box^m R_{2m}(x) + c_{m-1} \Box^{m-1} R_{2m}(x) + \ldots + c_1 \Box R_{2m}(x) + c_0 R_{2m}(x)) * u(x) = f(x) * R_{2m}(x)$$

By Lemma 2.2, Lemma 2.3(c), we obtain

$$(c_m \delta + c_{m-1} R_2(x) + c_{m-2} R_4(x) + \ldots + c_1 \Box R_{2(m-1)}(x) + c_0 R_{2m}(x)) * u(x) = f(x) * R_{2m}(x) \tag{3.4}$$

By Lemma 2.3(a), we obtain

$$R_4(x) = \frac{V^{\frac{4-n}{2}}}{K_n(4)} = \frac{V^{\frac{2-n}{2}}.V}{2(2+2-n)K_n(2)} = R_2(x). \frac{V}{2(4-n)}$$

Similarly,

$$R_6(x) = R_2(x). \frac{V^2}{2.4(4-n)(6-n)}$$

$$R_8(x) = R_2(x). \frac{V^3}{2.4.6(4-n)(6-n)(8-n)}$$

$$\vdots$$

$$R_{2m}(x) = R_2(x). \frac{V^{m-1}}{2.4 \ldots 2(m-1)(4-n)(6-n)\ldots(2m-n)}$$

Thus we obtain the function $W(x)$ of (3.3). Now $W(x)$ is continuous and infinitely differentiable in classical sense for n is odd. Since $R_2(x)$ is a tempered distribution with compact support, hence $W(x)R_2(x)$ also is tempered distribution with compact support and so $c_m R_0(x) + W(x)R_2(x)$. By Lemma 2.4(c), $c_m R_0(x) + W(x)R_2(x)$ has a unique inverse denoted by $(c_m R_0(x) + W(x)R_2(x))^{*-1}$.

Now (3.4) can be written as

$$(c_m R_0(x) + W(x)R_2(x)) * u(x) = f(x) * R_{2m}(x), \; R_0(x) = \delta$$

Convolving both sides by $(c_m R_0(x) + W(x)R_2(x))^{*-1}$, we obtain

$$u(x) = f(x) * R_{2m}(x) * (c_m R_0(x) + W(x)R_2(x))^{*-1}$$

Since $R_{2m}(x)$ is a unique by Lemma 2.2 and $(c_m R_0(x) + W(x)R_2(x))^{*-1}$ also a unique by Lemma 2.4(c), it follows that $u(x)$ is a unique weak solution of (3.1) with odd dimensional n. This completes the proof.

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REFERENCES

Donoghue, W.F. 1969. Distributions and Fourier transform. Academic Press, New York.

Kananthai, A. 1997. On the distribution related to the ultra-hyperbolic equations. Computational and Applied Mathematics, Elsevier Science Inc. 84:101-106.

Nozaki, Y. 1964. On Riemann-Liouville Integral of Ultra-hyperbolic Type. Kodai Mathematical Seminar Reports. 6(2):69-87.

Tellez, M.A. 1994. The distribution Hankel transform of Marcel Riesz's ultra-hyperbolic kernel. Studies in Applied Mathematics. Massachusetts Institute of Technology. Elsevier Science Inc. 93:133-162.

Trione, S.E. 1987. On Marcel Riesz's ultra-hyperbolic kernel. Trabajos de Matematica, 116.

Zemanian, A.H. 1965. Distribution and transform analysis. McGraw-Hill, New York.